7.2 Numerical Integration based on Interpolation

Idea: some functions are easy to integrate analytically, e.g. $f(x) = e^x$

$f(x) = \sin x$

others, like $f(x) = e^{-x^2}$ or $f(\theta) = \int_0^\pi \sin(x \sin\theta)\, d\theta$ may not be.

Goal: Want a method to approximate

$\int_a^b f(x)\,dx$ by using only $f(x_i)$, $i = 0, \dots, n$.

Reasonable approach: use polynomial interp.

to write $f(x) \approx p(x) = \sum_{i=0}^{n} f(x_i) l_i(x)$

Lagrange form of interp. poly.

Then "hope" that $\int_a^b f(x)\,dx \approx \int_a^b p(x)\,dx$

So $$\int_a^b f(x)\,dx \approx \int_a^b \sum_{i=0}^{n} f(x_i)\,\ell_i(x)\,dx$$

$$= \sum_{i=0}^{n} \int_a^b f(x_i)\,\ell_i(x)\,dx$$

exchange order of Σ & $\int$

$$= \sum_{i=0}^{n} f(x_i) \int_a^b \ell_i(x)\,dx$$

$f(x_i)$ is a number; $\int_a^b \ell_i(x)\,dx$ → call this A_i

$$= \sum_{i=0}^{n} A_i f(x_i)$$

In fact, you already know some examples of this

Trapezoid rule: when $n=1$ & $x_0=a, x_1=b$

we have $$\ell_0(x) = \frac{x_1 - x}{x_1 - x_0} = \frac{b-x}{b-a}$$

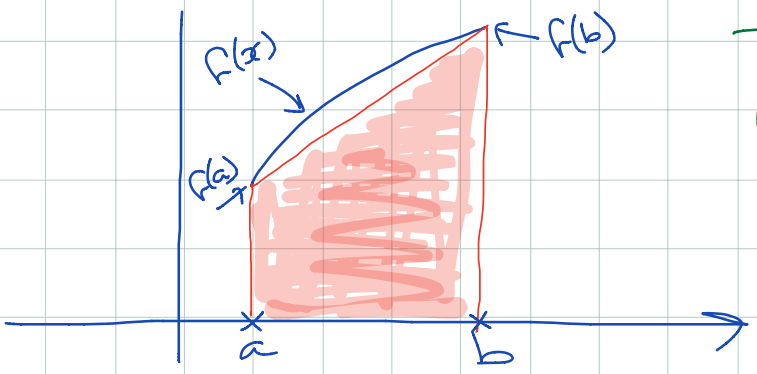
$$\ell_1(x) = \frac{x_0 - x}{x_0 - x_1} = \frac{x-a}{b-a}$$

$$\Rightarrow A_0 = \int_a^b \frac{b-x}{b-a}\,dx = \frac{b(b-a)}{b-a} - \frac{b^2-a^2}{2(b-a)} = b - \frac{b+a}{2}$$

$$A_0 = \frac{b-a}{2}$$

$$A_1 = \frac{b-a}{2} \text{ (similarly)}$$

$$\Rightarrow \int_a^b f(x)\,dx \approx \frac{b-a}{2}\left[f(a)+f(b)\right]$$



approximate the area under $f(x)$ by the area of the trapezoid.

Composite trapezoid rule: (1) Subdivide $[a,b]$ into n pieces using $a = x_0 < x_1 < x_2 < \cdots < x_n < b$

(2) use the trapezoid rule for each piece

$$\Rightarrow \int_a^b f(x) = \sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} f(x)\,dx$$

$$\approx \frac{(x_i - x_{i-1})}{2}\left[f(x_i) + f(x_{i-1})\right]$$

$$\Rightarrow$$

$$\int_a^b f(x)\,dx \approx \frac{1}{2}\sum_{i=1}^{n} (x_i - x_{i-1})\left(f(x_{i-1}) + f(x_i)\right)$$

Comp. Trap. rule with equal spacing

when all the subintervals of $[a,b]$ are the same length $h = x_i - x_{i-1}$

$$\int_a^b f(x)\,dx \approx \frac{h}{2}\sum_{i=1}^{n}\left(f(x_{i-1}) + f(x_i)\right)$$

$$= \frac{h}{2}\left[f(a) + 2\sum_{i=1}^{n-1} f(a+ih) + f(b)\right]$$

—X—

Back to the general, non-composite, case

Recall:

$$\int_a^b f(x)\,dx \approx \sum_{i=0}^{n} A_i f(x_i)$$

$$A_i \hookrightarrow \int_a^b \ell_i(x)\,dx$$

Since for polynomials of degree $\leq n$

$$f(x) = \sum_{i=0}^{n} f(x_i)\,\ell_i(x)$$ then our

integration formula is exact for poly. of deg $\leq n$.

This observation allows us to find the A_i's "easily" by the method of undetermined coefficients

Example: $n=2$, $[a,b]=[0,1]$ & $x_0=0, x_1=\frac{1}{2}, x_2=1$

$$\Rightarrow \int_a^b f(x) \approx A_0 f(0) + A_1 f(1) + A_2 f(2)$$

Formula is exact for poly. of degree ≤ 2

So $\int_0^1 \underbrace{1}_{f(x)} dx = A_0 + A_1 + A_2$ (integral $= 1$)

$\int_0^1 \underbrace{x}_{f(x)} dx = A_0 \cdot 0 + A_1 \cdot \frac{1}{2} + A_2$ (integral $= \frac{1}{2}$)

$\int_0^1 x^2 dx = A_0 \cdot 0 + A_1 \frac{1}{4} + A_2$ (integral $= \frac{1}{3}$)

3 eq'n & 3 unknowns $\Rightarrow A_1 = \frac{2}{3}$, $A_2 = \frac{1}{6} = A_0$

& $\int_0^1 f(x)dx \approx \frac{1}{6} f(0) + \frac{2}{3} f(\frac{1}{2}) + \frac{1}{6} f(1)$

Simpson's Rule

Repeat the same calculation but with arbitrary $[a,b]$ & $x_0 = a$, $x_1 = \frac{a+b}{2}$, $x_2 = b$

to get

$$\int_a^b f(x)dx \approx \frac{b-a}{6}\left(f(a) + 4f\left(\frac{b+a}{2}\right) + f(b)\right)$$

Simpson's rule

Remark: should be exact for poly. of degree ≤ 2

but is also exact for poly. of degree ≤ 3

Composite simpson's rule can also be used (see book)

Error Analysis

Want an expression for the error in numerical integration; that is, we want an expression for

$$\int_a^b f(x) - \sum_{i=0}^{N} A_i f(x_i)$$

Recall: $A_i = \int_a^b \ell_i(x)\,dx$ where $\ell_i(x)$ comes from the Lagrange interp. poly:

$$P(x) = \sum_{i=0}^{n} f(x_i)\,\ell_i(x)$$

Also: $f(x) - P(x) = \frac{1}{(n+1)!} f^{(n+1)}(\xi_x) \prod_{i=0}^{n} (x - x_i)$

So: $$\int_a^b f(x)\,dx - \sum_{i=0}^{n} A_i f(x_i) = \frac{1}{(n+1)!} \int_a^b f^{(n+1)}(\xi_x) \prod_{i=0}^{n} (x - x_i)\,dx$$

can't do much about this as it depends on the f we're numerically integrating. Reasonable to just use an upper bound

$$M = \max_{x \in [a,b]} \left| f^{(n+1)}(x) \right|$$

or $M \geq \dots$

Then $$\left| \int_a^b f(x)\,dx - \sum_{i=0}^{n} A_i f(x_i) \right| \leq \frac{M}{(n+1)!} \int_a^b \prod_{i=0}^{n} |x - x_i|\,dx$$

Previously we used the roots of the Chebyshev Poly. $T_{n+1}(x)$ as our nodes x_i to minimize $\prod_{i=0}^{n} (x - x_i)$. We saw that gives $\prod_{i=0}^{n} (x - x_i) = 2^{-n}$

Now, we want to minimize $\int_a^b \prod_{i=0}^{n} |x - x_i|\,dx$

Turns out, when $[a,b] = [-1,1]$, the choice of nodes

$$x_i = \cos\left(\frac{(i+1)\pi}{n+2}\right) \qquad i = 0, \dots, n$$

is optimal ← See Theorem 1 on page 487.

These are the roots of the polynomial

$$U_{n+1}(\cos\theta) = \frac{\sin((n+2)\theta)}{\sin\theta}$$

$$\Leftrightarrow U_{n+1}(x) = \frac{\sin\left((n+2)\cos^{-1}x\right)}{\sin\left(\cos^{-1}x\right)}$$

→ Chebyshev Polynomial of the 2nd kind

Remark: Just as $T_{n+1}(x)$ was not monic, $U_{n+1}(x)$ is not monic. However, $2^{-(n+1)}U_{n+1}$ is monic.

So $2^{-(n+1)}U_{n+1} = \prod_{i=0}^{n}(x - x_i)$

where $x_i = \cos\frac{(i+1)\pi}{n+2}$

Moreover, $\int_{-1}^{1} \left|\prod_{i=0}^{n} x - x_i\right| dx = 2^{-n}$

So, with this choice of nodes:

$$\left| \int_{-1}^{1} f(x) - \sum_{i=0}^{n} A_i f(x_i) \right| \leq \frac{M}{(n+1)!} 2^{-n}$$

→ Error in numerical integration when using zeros of U_{n+1} as nodes.

Remark: Just as with $T_n(x)$, $U_n(x)$ can be generated recursively

via: $U_0(x) = 1$, $U_1(x) = 2x$

$U_{n+1} = 2x U_n - U_{n-1}$ when $n \geq 1$

Can you prove it?

Remark: $T_n'(x) = n U_{n-1}(x)$

Can you prove it?

HW 7.2 # 1, 4, 5, 8, 12